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EQUICONVERGENCE AND EQUISUMMABILITY OF JACOBI SERIES

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Abstract. In this paper we give some results concerning the equiconvergence and equisummability of series in Jacobi polynomials.

1. Jacobi polynomials and series in the complex plane.

Let $\alpha > -1$ and $\beta > -1$. The polynomials $\left\{P_n^{(\alpha,\beta)}(z)\right\}_{n=0}^{+\infty}$ defined by the equalities

$$P_n^{(\alpha,\beta)}(z) = \binom{n+\alpha}{n} F\left(-n, n+\alpha+\beta+1, \alpha+1; \frac{1-z}{2}\right),$$

$$z \in \mathbb{C}, \quad n = 0, 1, 2, \ldots,$$

where $F(a, b, c; \zeta)$ is the Gauss hypergeometric function, are called *Jacobi polynomials with parameters* α *and* β.

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There exists a unique complex function h, holomorhpic in the region $G = \mathbb{C}\backslash[-1,+1]$, such that $h^2(z) = z^2 - 1$ when $z \in G$ and $h(x) > 0$ when $x > 1$. The value of this function at any point $z \in G$ is denoted by $\sqrt{z^2-1}$. The function w defined in G as

$$w(z) = z + \sqrt{z^2-1}$$

is holomorphic in G. Moreover, $w(z) \neq 0$ and $[w(z) + (w(z))^{-1}]/2 = z$ when $z \in G$. Therefore, $w(z)$ may be considered as an inverse of the Zhukovsky function. As it is well known, the latter one is univalent in the region $D = \{w : |w(z)| > 1\}$ and maps it onto G. Hence, the function w maps G onto D. The function w is meromorphic in the region $G = \overline{\mathbb{C}}\backslash[-1,+1]$ and $\lim\limits_{z\to\infty} w(z) = \infty$.

G. Szegö gives an asymptotic expansion for Jacobi polynomials in the region G [4, Theorem 8.21.1]. From this expansion it follows the asymptotic formula ($n \geq 1$, $p = 0, 1, 2, \dots$)

$$P_n^{(\alpha,\beta)}(z) = \psi(z) n^{-1/2} (w(z))^n \left\{ \sum_{k=0}^{p} \varphi_k(z) n^{-k} + \varphi_{n,p}(z) \right\}, \tag{1.1}$$

where $\psi(z) \neq 0$, $\{\varphi_k(z)\}_{k=0}^{+\infty}$, $\{\varphi_{n,p}(z)\}_{n=1}^{+\infty}$ ($p = 0, 1, 2, \dots$) are a complex functions holomorphic in the region G and

$$\varphi_{n,p}(z) = O(n^{-p-1}) \quad (n \to \infty) \tag{1.2}$$

uniformly on every compact subset of G. In particular, $\varphi_0(z) \equiv 1$.

If $p = 0$, then it follows from (1.1) and (1.2) that [3, (II.3.1)]

$$P_n^{(\alpha,\beta)}(z) = \psi(z) n^{-1/2} (w(z))^n \{1 + \varphi_{n,0}(z)\}, \tag{1.3}$$

where $\varphi_{n,0}(z) = O(n^{-1})$ ($n \to \infty$) uniformly on every compact subset of G.

Suppose that $r > 1$. Denote by $\gamma(r)$ the image of the circle $C(0,r) = \{z \in \mathbb{C} : z = r \exp i\theta,\ 0 \leq \theta \leq 2\pi\}$ by the Zhukovsky transformation. As it is well known, $\gamma(r)$ is the ellipse with focuses at the points -1 and $+1$ and semiaxes $(r + r^{-1})/2$ and $(r - r^{-1})/2$.

Let $E(r)$ be the interior of $\gamma(r)$. Further, we denote $E(\infty) = \mathbb{C}$, $E(1) = \emptyset$, $E^*(r) = \overline{\mathbb{C}}\backslash\overline{E(r)}$ ($1 < r < \infty$), $E^*(\infty) = \emptyset$ and $E^*(1) = G$.

The series of the kind

$$\sum_{n=0}^{+\infty} a_n P_n^{(\alpha,\beta)}(z) \tag{1.4}$$

has the name *Jacobi series.*

Theorem 1.1 [3, (IV.1.1), (a)]. *If the series* (1.4) *converges at a point* $z_0 \in G$*, then it is absolutely uniformly convergent on every compact subset of the region* $E(r)$ *with* $r = |w(z_0)|$.

We remark that in the proof of this theorem it is used the asymptotic formula (1.3).

Theorem 1.2 [3, (IV.1.1), (b)]. *If*

$$r = \max\left\{1, \left(\lim_{n\to+\infty} \sup |a_n|^{1/n}\right)^{-1}\right\}, \tag{1.5}$$

then the series (1.4) *is absolutely uniformly convergent on every compact subset of the domain* $E(r)$ *and diverges in* $E^*(r)$.

The equality (1.5) can be regarded as a *formula of Cauchy-Hadamard type* for the Jacobi series. If $r > 1$, then the sum of the series (1.4) is a complex function, holomorphic in the domain $E(r)$.

2. (C, δ)-summability of a series. Let us remind that a series

$$\sum_{n=0}^{+\infty} u_n \tag{2.1}$$

is said to be (C, δ)-summable for $\delta > -1$, if there exists

$$\sigma = \lim_{n\to+\infty} \left(A_n^\delta\right)^{-1} \sum_{k=0}^{n} A_{n-k}^{\delta-1} s_k,$$

where $A_k^\delta = \Gamma(k+\delta+1)/[\Gamma(k+1)\Gamma(\delta+1)]$ ($\delta > -1$; $k = 0, 1, \dots$) and $\{s_k\}_{k=0}^{+\infty}$ are the partial sums of the series (2.1).

Usually the complex number σ is called (C, δ)-sum of the series (2.1). Every convergent series is (C, δ)-summable for every $\delta > 0$ and its (C, δ)-sum is equal to its sum in the usual sense.

3. Equisummability of Jacobi series. Let us to consider a series of kind

$$\sum_{n=1}^{+\infty} a_n n^{-1/2}(w(z))^n. \tag{3.1}$$

In the following proposition we give a relation between the series (1.4) and (2.1).

Theorem 3.1. *Let $\delta > -1$ and $z_0 \in G$. Then the series* (1.4) *is* (C, δ)-*summable at z_0 if and only if the series* (2.1) *is* (C, δ)-*summable at z_0.*

Proof. Let p be an integer such that $p > \delta + 1$. Further, we assume that $a_0 = 0$.

Suppose that the series (1.4) is (C, δ)-summable for $z = z_0$. Applying [2, Theorem 46], we obtain that

$$a_n P_n^{(\alpha,\beta)}(z_0) = o(n^\delta) \quad (n \to \infty). \tag{3.2}$$

We define the numbers $\mu_1, \mu_2, \dots, \mu_p$ by the following equalities

$$\mu_1 = \varphi_1(z_0)$$

$$\mu_2 = \varphi_2(z_0) - \mu_1\varphi_1(z_0)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\mu_p = \varphi_p(z_0) - \sum_{j=1}^{p-1} \varphi_{p-j}(z_0)\mu_j.$$

If

$$b_n = a_n P_n^{(\alpha,\beta)}(z_0)\{1 - \mu_1 n^{-1} - \dots - \mu_p n^{-p}\},$$

then from [2, Theorem 74] it follows that the series $\sum\limits_{k=1}^{+\infty} b_n$ is (C, δ)-summable. Using the asymptotic formulas (1.1) and (1.2), we conclude that

$$b_n = \psi(z_0) a_n n^{-1/2}(w(z_0))^n\{1 + d_{n,p}(z_0)\},$$

where $d_{n,p}(z_0) = O(n^{-p-1})$ $(n \to \infty)$. Obviously, $\psi(z_0) \neq 0$ and $P_n^{(\alpha,\beta)}(z_0) \neq 0$ $(n = 1, 2, \dots)$ Then, from (3.2), (1.1) and (1.2) it follows that

$$a_n \psi(z_0) n^{-1/2}(w(z_0))^n = o(n^\delta) \quad (n \to \infty).$$

Further, we have

$$q_n = \psi(z_0) a_n n^{-1/2} (w(z_0))^n d_{n,p}(z_0) = o(n^\delta) O(n^{-p-1}) \quad (n \to \infty).$$

Since $p > \delta + 1$ then

$$q_n = O(n^{-2}) \quad (n \to \infty).$$

Hence, the series $\sum\limits_{k=1}^{+\infty} q_n$ is convergent. Applying [3, Theorem 45] we deduce that this series is (C,δ)-summable. Then, from the equalities

$$a_n n^{-1/2} (w(z_0))^n = b_n / \psi(z_0) - q_n \quad (n = 1, 2, \dots)$$

it follows that the series (3.1) is (C,δ)-summable.

Now, let us assume that the series (3.1) is (C,δ)-summable for $z = z_0$. Then

$$a_n n^{-1/2} (w(z_0))^n = o(n^\delta) \quad (n \to \infty).$$

This asymptotic formula and the asymptotic formula (1.2) yield

$$c_n = \psi(z_0) a_n n^{-1/2} (w(z_0))^n d_{n,p}(z_0) = o(n^\delta) O(n^{-p-1}).$$

Since $p > \delta + 1$, then $c_n = O(n^{-2})$. Hence the series $\sum\limits_{k=1}^{+\infty} c_n$ converges. According to [3, Theorem 45] it is (C,δ)-summable.

The series of the kind

$$\sum_{n=1}^{+\infty} a_n (w(z_0))^n n^{-s-1/2},$$

where $s = 1, 2, \dots, p$, is (C,δ)-summable by Theorem 45 from [3]. Then, from the equalities $(n = 1, 2, \dots)$

$$a_n P_n^{(\alpha,\beta)}(z_0) = \psi(z_0) a_n n^{-1/2} (w(z_0))^n + \psi(z_0) \sum_{s=1}^{p} a_n (w(z_0))^n \varphi_s(z_0) n^{-s-1/2} + c_n$$

it follows that the series (3.1) is (C,δ)-summable for $z = z_0$. Thus Theorem 3.1 is proved. $\square$

Having in mind the proof of Theorem 3.1 it is easy to establish the following result:

Theorem 3.2. *If $\delta > -1$ and K is a compact subset of the region G, then the series* (1.4) *is uniformly* (C, δ)*-summable on K if and only if the series* (3.1) *is uniformly* (C, δ)*-summable on K.*

The following statement is a corollary of Theorem 3.1.

Theorem 3.3. *If $z_0 \in G$, then the series* (1.4) *converges for $z = z_0$ if and only if the series* (3.1) *converges for $z = z_0$.*

Theorems 3.1 and 3.3 can be called *Theorems for equisummability and equiconvergence* of Jacobi series, respectively.

4. Fatou-Riesz theorem for Jacobi series. Let

$$\sum_{n=0}^{+\infty} a_n t^n \tag{4.1}$$

be a power series with finite and different from zero radius r of convergence and $g(t)$ be a sum of (4.1) in the region $\operatorname{int}\{C(0,r)\}$. It is well known that if the coefficients $\{a_n\}_{n=0}^{+\infty}$ satisfy certain additional conditions, there is a relation between the regular points of the function $g(t)$ on the circumference $C(0,r)$ and the behaviour of (4.1) on the same circumference. The following result holds [6, (9.21)].

Theorem 4.1 (Fatou-Riesz)**.** *If the power series* (4.1) *has a finite radius r of convergence and $a_n r^n = o(n^\delta)$ $(n \to +\infty)$ for some $\delta > -1$, then the series* (4.1) *is uniformly* (C, δ)*-summable on every close arc of $C(0,r)$, all points of which are regular for the function $f(t)$.*

V. Yatsun in [5] described the relations between the singular points of Jacobi series (1.5) and those of the corresponding power series (4.1). In particular it follows from his main statement:

Theorem 4.2. *Let $1 < r = \lim_{n\to+\infty} \sup |a_n|^{1/n} < +\infty$ and $z_0 \in \gamma(r)$ be a regular point for the sum $f(z)$ of* (1.4) *in $E(r)$. Then $t_0 = w(z_0) \in C(0,r)$ is a regular point for the sum $g(t)$ of* (4.1)*.*

Using Theorems 4.1 and 4.2 we are going to prove a theorem for the Jacobi series, which is similar to Theorem 4.1.

Theorem 4.3 [1]**.** *Suppose that r from* (1.4) *satisfies the inequalities $1 < r < +\infty$ and $a_n r^n = o(n^\delta)$ $(n \to +\infty)$ for some $\delta > -1/2$. Then the series*

(1.4) *is uniformly* $(C, \delta - 1/2)$*-summable on every close arc* $\gamma_1 \subset \gamma(r)$*, all points of which are regular for the sum* $f(z)$ *of* (1.4).

P r o o f. Let $\gamma_1^* = \{t : t = w(z), z \in \gamma_1\}$. Obviously, γ_1^* is a close arc of the circumference $C(0, r)$. According to Theorem 4.2 all point of this arc are regular for the sum $g(t)$ of (4.1). Then, using Theorem 4.1 the series (4.1), we obtain that this series is uniformly (C, δ)-summable on the arc γ_1^*. Then, the series

$$\sum_{n=1}^{+\infty} a_n t^n$$

is also uniformly (C, δ)-summable on the arc γ_1^*. Using [2, Theorem 76], we obtain that the series

$$\sum_{n=1}^{+\infty} a_n n^{-1/2} t^n$$

is uniformly $(C, \delta - 1/2)$-summable on the arc γ_1^*. Hence, the series (3.1) is uniformly $(C, \delta - 1/2)$-summable for $z \in \gamma_1$. According Theorem 3.2 the series (1.5) is uniformly $(C, \delta - 1/2)$-summable on the arc γ_1. Thus Theorem 4.3 is proved. □

Remark. The proof of Theorem 4.3 is considerably shorter than its proof given in [1].

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